

A. Standard exercises

14.1 Consider the hyperboloid \mathcal{H} described as the graph of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = x^2 - y^2$.

- (a) Find a global parametrization $\psi : \mathbb{R}^2 \rightarrow \mathcal{H}$ of \mathcal{H} . With respect to this parametrization, compute the associated tangent vectors b_1, b_2 , as well as the unit normal n .
- (b) Thinking of the normal n as a function from \mathcal{H} to \mathbb{R}^3 , compute its differential dn_p at every $p \in \mathbb{R}^2$. Verify that (as expected), the image of dn_p is in the span of b_1, b_2 . Expressing the columns of dn_p with respect to the basis b_1, b_2 , obtain an expression for the matrix of the shape operator $L_p : T_p\mathcal{H} \rightarrow T_p\mathcal{H}$.
- (c) Verify that the above expression is the same that you would get by computing L via the formula $L = -G^{-1}H$, where G is the matrix of the metric tensor and H is the matrix of the second fundamental form (note that, in general, the latter procedure is faster and less prone to mistakes).

14.2 Let $\gamma(s) = (r(s), z(s))$ be a naturally parametrized C^2 curve in the plane with $r(s) > 0$ and let $\psi(s, \theta) = (r(s) \cos \theta, r(s) \sin \theta, z(s))$ be the parametrization of the corresponding surface of revolution around the Oz axis. Compute the Christoffel symbols Γ_{ijk} and Γ_{ij}^k .

14.3 (a) Let $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a **simple, closed** C^2 curve in the oriented plane. Assume that its oriented curvature, denoted k , satisfies $0 \leq k(t) \leq C$ for all $t \in [a, b]$.
Prove that the length ℓ of γ satisfies

$$\ell \geq \frac{2\pi}{C}.$$

(b) Can this bound be achieved? I.e. does there exist a simple closed C^2 curve γ in \mathbb{R}^2 such that

$$k(t) = \frac{2\pi}{\ell(\gamma)} \quad \text{for all } t?$$

14.4 Let $\mathcal{T} \subset \mathbb{R}^3$ be the torus obtained by rotating the circle of radius 1 and center $(a, 0, 0)$ in the Oxz -plane around the Oz -axis (assume $a > 1$, so that the rotated figure doesn't self intersect).

(a) The first part of this exercise is purely topological. Convince yourself that, topologically, the torus can be obtained from the following gluing process: Start from a rectangle (with consecutive sides a, b, c, d , and glue first the sides a and c (the resulting shape should look like a cylinder) and then glue the remaining two sides. In the resulting construction, we have identified all four vertices of the original square into one, and the four original edges of the square into two (becoming two intersecting circles on the torus). What would be the Euler characteristic of the torus?

- (b) Let's now focus on the geometric surface \mathcal{T} in \mathbb{R}^3 . Find a parametrization and compute its associated metric tensor.
- (c) Compute the area of the surface \mathcal{T} .
- (d) Compute the Gaussian curvature K of \mathcal{T} (express K as a function of the parameters of the parametrization given in (a)).
- (e) Compute the integral

$$\iint_{\mathcal{T}} K \, dA.$$

Verify the Gauss–Bonnet formula.

B. Bonus exercise

14.5 The Laplace–Beltrami operator on a C^2 surface S is defined as followed: For any C^2 function $f : S \rightarrow \mathbb{R}$ and for any C^2 local parametrization $\psi : \Omega \rightarrow S$, we have

$$\Delta_g f \doteq \sum_{i,j=1}^2 g^{ij} \partial_i \partial_j f - \sum_{i,j,k=1}^2 g^{ij} \Gamma_{ij}^k \partial_k f,$$

where g_{ij} are the components of the metric tensor G of the parametrization, g^{ij} are the components of G^{-1} and Γ_{ij}^k are the Christoffel symbols. One can show that Δ_g is independent of the choice of parametrization (this is merely a long computation, based on how the metric tensor, Christoffel symbols etc change under a change of coordinates); we won't pursue this here.

- (a) Show that, in the special case when $S = \mathbb{R}^2$ parametrized with the usual Cartesian coordinates, then Δ_g is the usual Laplace operator.
- (b) Assume that the parametrization $\psi : \Omega \rightarrow S$ is *conformal*, i.e. the metric tensor takes the form

$$g_{ij} = f \cdot \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, while $f : \Omega \rightarrow (0, +\infty)$ is a C^2 function. Compute the Christoffel symbols.

- (c) In the case of part (b) above, show that the coordinate functions $x_1, x_2 : \psi(\Omega) \rightarrow \mathbb{R}$ are harmonic, i.e. satisfy

$$\Delta_g x_i = 0.$$

Recall that, when $\psi : \Omega \rightarrow S$ is a parametrization, the coordinates $x_i : \psi(\Omega) \subset S \rightarrow \mathbb{R}$ are simply the pushforwards of the Cartesian coordinates on Ω , i.e. $x_i = u_i \circ \psi^{-1}$. In particular, the map $(x_1, x_2) : \psi(\Omega) \subset S \rightarrow \Omega \subset \mathbb{R}^2$ is the inverse of ψ .

Remark. In fact, the above statement can be also reversed: If x_1, x_2 are two *conjugate* harmonic functions on S (i.e. their gradients are perpendicular), then the corresponding parametrization is conformal. In particular, any surface in R^3 has locally a conformal parametrization (this is an important step for the proof of the uniformization theorem for compact surfaces). This statement is false for higher dimensional submanifolds.